IOPscience iopscience.iop.org

Home Search Collections Journals About Contact us My IOPscience

Collinear topological monopoles and topological lines

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

1978 J. Phys. A: Math. Gen. 11 173

(http://iopscience.iop.org/0305-4470/11/1/018)

View the table of contents for this issue, or go to the journal homepage for more

Download details:
IP Address: 129.252.86.83
The article was downloaded on 30/05/2010 at 14:10

Please note that terms and conditions apply.

J. Phys. A: Math. Gen., Vol. 11, No. 1, 1978. Printed in Great Britain. © 1978

Collinear topological monopoles and topological lines

H Schiff

Institute of Theoretical Physics, Department of Physics, University of Alberta, Edmonton, Alberta, Canada T6G 2J1

Received 19 July 1977

Abstract. Starting with the magnetic field of an arrangement of magnetic monopoles on a line we construct an appropriate unit isovector and gauge fields. We find examples of monopoles connected by topological lines where the total pole strength must be integral but the individual monopoles need not be integers.

1. Introduction

The topology of magnetic charges is generally discussed within the framework of the homotopy class of point singularities. Now it is known that in addition to point singularities there are physical systems in which topological line singularities play an important role, such as disclinations in nematic liquid crystals and Néel lines in ferromagnetic crystals. The question thus arises whether similar topological configurations may exist in systems involving magnetic charges. The purpose of this paper is to show that this is indeed the case. Specifically, given an appropriate arrangement of magnetic charges we find that they may be connected to an extended structure consisting of topological line singularities. In § 2 we obtain a simple expression for the magnetic field of a collinear system of magnetic charges. In § 3 we construct the unit isovector for the magnetic field and the corresponding gauge fields, and concluding remarks are made in § 4.

2. The magnetic field

To illustrate our construction we confine ourselves to a spontaneously broken SO(3) gauge theory involving a triplet of Yang–Mills fields A^a_μ and a triplet of Higgs fields ϕ^a. One can then construct an electromagnetic tensor ('t Hooft 1974) with the decomposition (Arafune *et al* 1975)

$$F_{\mu\nu} = M_{\mu\nu} + H_{\mu\nu} \tag{2.1}$$

where

$$\begin{aligned} M_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu \\ B_\mu &= A^a_\mu v^a \\ H_{\mu\nu} &= -\frac{1}{e}\epsilon_{abc} v^a\, \partial_\mu v^b\, \partial_\nu v^c \end{aligned} \tag{2.2}$$

and the unit isovector $v^a = \phi^a/(\phi^b\phi^b)^{1/2}$.

We will show that (2.2) can be expressed in a compact and suggestive way, and this in turn will lead to a simple construction of unit isovectors and gauge fields for collinear monopoles.

We parametrise the unit sphere $v^a v^a = 1$ with the two variables ξ and η, so that $v^a = v^a(\xi, \eta)$. Making use of the antisymmetry of ϵ_{abc} we can write (in units of $-e = 1$)

$$H_{\mu\nu} = T(\xi, \eta)(\partial_\mu \xi \partial_\nu \eta - \partial_\nu \xi \partial_\mu \eta) \tag{2.3}$$

where

$$T(\xi, \eta) = \epsilon_{abc} v^a \partial_\xi v^b \partial_\eta v^c. \tag{2.4}$$

Incidentally, from (2.3) it follows that the invariant $H^{\mu\nu} H^*_{\mu\nu} = 0$, where the dual tensor $H^*_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} H^{\rho\sigma}$.

In our expression for $H_{\mu\nu}$ in (2.3), a natural choice for ξ and η is the isopolar angles Φ and Θ, where $\Phi = \tan^{-1}(v^2/v^1)$ and $\Theta = \cos^{-1} v^3$, i.e.

$$v^1 = \sin\Theta \cos\Phi \qquad v^2 = \sin\Theta \sin\Phi \qquad v^3 = \cos\Theta. \tag{2.5}$$

It is a simple matter to show that (2.4) now becomes

$$T(\Phi, \Theta) = -\sin\Theta \qquad \text{or} \qquad T(\Phi, \cos\Theta) = 1. \tag{2.6}$$

Thus for any unit isovector, (2.5) and (2.6) lead to the expression

$$H_{\mu\nu} = (\partial_\mu \Phi\, \partial_\nu \cos\Theta - \partial_\nu \Phi \partial_\mu \cos\Theta). \tag{2.7}$$

Since $T(\xi, \eta)$ is invariant under rotation of $\boldsymbol{v}$, the components, v^a, in (2.5) can be cyclically permuted without changing the results (2.6) and (2.7).

3. Isovectors and gauge fields

Before dealing with the case of collinear monopoles it will be useful to recall the results ('t Hooft 1974, Polyakov 1974) for a single unit monopole for which $\Phi = \phi$ and $\Theta = \theta$, where ϕ and θ are the polar coordinates in configuration space. The isovector is

$$v^1 = \sin\theta \cos\phi \qquad v^2 = \sin\theta \sin\phi \qquad v^3 = \cos\theta \tag{3.1}$$

and (2.7) becomes

$$H_{ij} = \partial_i \phi \partial_j \cos\theta - \partial_j \phi \partial_i \cos\theta. \tag{3.2}$$

It is readily checked that (3.2) does in fact give

$$H_{ij} = \epsilon_{ijk} x_k / r^3.$$

An Abelian field for the monopole is

$$B_i = (1 - \cos\theta)\partial_i \phi = \frac{\epsilon_{i3k} x_k}{r(r + x_3)} \tag{3.3}$$

the familiar Dirac string along the negative x_3 axis. Indeed the curl of (3.3) is just

$$\partial_i B_j - \partial_j B_i = H_{ij} + G_{ij} \tag{3.4}$$

where H_{ij} is given by (3.2) and

$$G_{ij} = (1 - \cos\theta)(\partial_i \partial_j - \partial_j \partial_i)\phi \tag{3.5}$$

i.e.

$$G_{12}=\begin{cases}0, & x_3>0\\ 4\pi\delta(x_1)\delta(x_2), & x_3\leqslant 0\end{cases}$$
$$G_{i3}=0 \tag{3.6}$$

so that (3.4) and (3.6) correspond to the well known behaviour of the Dirac potential, i.e. curl B consists of the Coulomb field of the monopole plus a δ function contribution to H_{12} on the string. We consider now a number of monopoles on the x_3 axis with strengths g_α, the index labelling, say, the position on the axis beginning at the top with $\alpha=1$; then an appropriate Abelian field for the system is evidently

$$B_i=\sum_\alpha g_\alpha(1-\cos\theta_\alpha)\partial_i\phi. \tag{3.7}$$

From g_α we get the magnetic field contribution

$$H_{ij}^\alpha=g_\alpha(\partial_i\phi\,\partial_j\cos\theta_\alpha-\partial_j\phi\,\partial_i\cos\theta_\alpha), \tag{3.8}$$

so that the total magnetic field is

$$H_{ij}=\sum_\alpha H_{ij}^\alpha=\left[\partial_i\phi\,\partial_j\Big(\sum_\alpha g_\alpha\cos\theta_\alpha\Big)-\partial_j\phi\,\partial_i\Big(\sum_\alpha g_\alpha\cos\theta_\alpha\Big)\right]. \tag{3.9}$$

Suppose now that we can define isopolar angles Φ and Θ in such a way that (3.9) can be expressed in the form (2.7) and the corresponding unit isovector (2.5) is single valued. This isovector can be transformed into (0, 0, 1) by the gauge transformation

$$\omega(\Theta,\Phi)=\mathrm{e}^{-\mathrm{i}\Phi I_3}\,\mathrm{e}^{\mathrm{i}\Theta I_2}\,\mathrm{e}^{\mathrm{i}\Phi I_3}. \tag{3.10}$$

Ignoring for the moment the gauge field solutions of the classical Yang–Mills equations associated with (2.5), we obtain under (3.10) the fields (from $\mathrm{i}\partial_i\omega\omega^{-1}$)

$$\begin{aligned}A_i'^1&=(\sin\Phi\,\partial_i\Theta+\sin\Theta\cos\Phi\,\partial_i\Phi)\\ A_i'^2&=(-\cos\Phi\,\partial_i\Theta+\sin\Theta\sin\Phi\,\partial_i\Phi)\\ A_i'^3&=(1-\cos\Theta)\,\partial_i\Phi.\end{aligned} \tag{3.11}$$

With $v^{a'}=(0,0,1)$ we have the Abelian field

$$A_i'^3=(1-\cos\Theta)\,\partial_i\Phi \tag{3.12}$$

which is designed to contain the strings for the monopoles. A necessary condition for this requirement is readily obtained. We observe that (3.9) is invariant under the transformation

$$\phi\to m\phi \qquad \text{and} \qquad \sum_\alpha g_\alpha\cos\theta_\alpha\to\frac{1}{m}\Big(\sum_\alpha g_\alpha\cos\theta_\alpha+k\Big)$$

where m and k are suitable constants which permit the identification

$$\Phi=m\phi \qquad \cos\Theta=\frac{1}{m}\Big(\sum_\alpha g_\alpha\cos\theta_\alpha+k\Big). \tag{3.13}$$

Clearly, single-valuedness requires that m in (3.13) be an integer. Furthermore if

(3.12) is equated to (3.7) we require, using (3.13), that

$$\left[1-\frac{1}{m}\left(\sum_\alpha g_\alpha \cos\theta_\alpha + k\right)\right]\partial_i m\phi = \sum_\alpha g_\alpha(1-\cos\theta_\alpha)\,\partial_i\phi \qquad (3.14)$$

so that

$$m-k=\sum_\alpha g_\alpha = n \qquad (3.15)$$

where n is the total magnetic charge. Now if $n\neq 0$, the asymptotic behaviour of v^a must conform to that of a pole of strength n at the origin, therefore

$$m=n,\ k=0, \qquad \text{for } n\neq 0. \qquad (3.16)$$

We note that requirement (3.14) is a physical one in the sense that the magnetic field (3.9) and the corresponding Abelian field (3.12) yield the same flux on integration. Condition (3.16) on the other hand invokes the appropriate *topological* behaviour for a monopole of strength n.

Further elaborations on the above construction are more easily examined by dealing with specific cases. Consider, for example, two monopoles of strengths g_1 and g_2 with $g_1>0$. First let $g_1+g_2=0$. From (3.13) and (3.15) we have

$$\Phi = m\phi \qquad \cos\Theta = 1+\frac{1}{m}(g_1\cos\theta_1+g_2\cos\theta_2).$$

On the x_3 axis between the monopoles

$$\cos\Theta = 1+\frac{1}{m}(-g_1+g_2)=1-\frac{2g_1}{m},$$

so that we must have the integer $m\geqslant g_1$. Only for $m=g_1$ is $\cos\Theta=-1$, otherwise $|\cos\Theta|<1$ and this leads to a singular behaviour for the unit isovector on the x_3 axis between the monopoles. This region is thus a topological line singularity.

If $g_1+g_2=n>0$, we have from (3.13) and (3.15)

$$\Phi = n\phi \qquad \cos\Theta = \frac{1}{n}(g_1\cos\theta_1+g_2\cos\theta_2).$$

On the x_3 axis between the monopoles $\cos\Theta=(-g_1+g_2)/(g_1+g_2)$ so that $|\cos\Theta|>1$ if g_1 and g_2 have opposite signs, while for any g_1 and g_2 of the same sign $|\cos\Theta|<1$ and the line joining the monopoles is topologically singular. This is true even when g_1 and g_2 are equal and integral.

The above considerations apply also of course to more than two monopoles. The case of three monopoles illustrates the interesting result that the *arrangement* of the monopoles may be significant. It is easy to see, for example, that the three monopoles $g_1=n$, $g_2=-n$, $g_3=n$, in that order, (the integral values are chosen for simplicity) lead to $\cos\Theta=(\cos\theta_1-\cos\theta_2+\cos\theta_3)$ so that $|\cos\Theta|\leqslant 1$ everywhere (this system is free of topological line singularities). On the other hand, the arrangement $g_1=n$, $g_2=n$, $g_3=-n$ gives $\cos\Theta=(\cos\theta_1+\cos\theta_2-\cos\theta_3)$ which has the value -3 on the x_3 axis between g_2 and g_3. Hence this arrangement is not a possible one.

The gauge fields associated with (2.5) are easily obtained (Arafune *et al* 1975). In the gauge where $v'^a=(0,0,1)$ we choose $A_i'^1=A_i'^2=0$, $A_i'^3=(1-\cos\Theta)\,\partial_i\Phi$ and carry out the gauge transformation with $\omega^{-1}(\Theta,\Phi)=\omega(-\Theta,\Phi)$, from (3.10). This leads of course to the v^a as in (2.5) and one finds from $A_i^a \tau_a/2 = \omega^{-1}(A_i'^a \tau_a/2)\omega + i\, \partial_i\omega^{-1}\omega$, that

$$A_i^1 = -\sin\Phi\, \partial_i\Theta - \sin\Theta\cos\Theta\cos\Phi\, \partial_i\Phi$$

$$A_i^2 = \cos\Phi\, \partial_i\Theta - \sin\Theta\cos\Theta\sin\Phi\, \partial_i\Phi$$

$$A_i^3 = \sin^2\Theta\, \partial_i\Phi$$

or, simply,

$$A_i^a = \epsilon^{abc} v_b\, \partial_i v_c. \tag{3.17}$$

Evidently $v^a A_i^a = 0$, as expected, and one finds that (3.17) diverges only at the positions of the point or line topological singularities.

4. Discussion

We have shown classically in a non-Abelian gauge theory that if an arrangement of monopoles on a line is considered as a single system then only the total pole strength is required to be integral and the topological behaviour is characterised by localised line singularities. Although one may expect that such line singularities connecting integer poles could be removed by an appropriate gauge transformation this is not possible for non-integer poles. We have then a form of 'topological confinement', forbidding their complete separation.

In this paper we have only considered certain *topological* properties of collinear monopoles. The possibility of finite energy solutions for such systems is still to be determined; in this connection however it is of interest to note (Weinberg and Guth 1976) that (3.17) is a necessary requirement for the asymptotic form of the gauge fields.

Acknowledgments

The author would like to thank R Torgerson and P Freund for helpful discussions. This work was supported in part by the National Research Council of Canada.

References

Arafune J, Freund P G O and Goebel C J 1975 *J. Math. Phys.* **16** 433
't Hooft G 1974 *Nucl. Phys.* B **79** 276
Polyatov A M 1974 *JETP Lett.* **20** 194
Weinberg E J and Guth A H *Phys. Rev.* D **14** 1660